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Non-differentiable embedding of Lagrangian systems and partial differential equations

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ABSTRACT

We develop the non-differentiable embedding theory of differential operators and Lagrangian systems using a new operator on non-differentiable functions. We then construct the corresponding calculus of variations and we derive the associated non-differentiable Euler–Lagrange equation, and apply this formalism to the study of PDEs. First, we extend the characteristics method to the non-differentiable case. We prove that non-differentiable characteristics for the Navier–Stokes equation correspond to extremals of an explicit non-differentiable Lagrangian system. Second, we prove that the solutions of the Schrödinger equation are non-differentiable extremals of the Newton's Lagrangian.

Contents

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Au sujet du principe de moindre quantité d'action, je pense pour ma part qu'on peut le considérer
comme la clé universelle de tous les problèmes, tant statiques que dynamiques,
pour les questions relevant du mouvement des corps
– quels que soit leur nombre et quelle que soit la manière dont ils sont liés entre eux –
soit de l'équilibre et du mouvement des fluides quelconques...

Lettre de Lagrange à Euler du 17 Mai 1756

0. Introduction

This paper is a contribution to the general program of embedding theories of dynamical systems [6]. Following our previous work on the stochastic embedding theory developed with Darses [8,9] and the fractional embedding theory [7], we define the non-differentiable embedding of differential operators and equations using the quantum calculus [13].

The need for such a theory comes from physics. In 1992, Nottale introduced the theory of scale-relativity without the hypothesis of space–time differentiability [14,15]. As a consequence, at the microscopic scale one must take into account this loss of differentiability. Natural trajectories are everywhere non-differentiable. Nottale studies the effect of the loss of differentiability on classical equations of mechanics. More precisely, given an ordinary differential equation (ODE), Nottale's proposal can be mathematically translated as follows:

1. The ODE is the restriction over differentiable solutions of a more general "differential" equation on non-differentiable functions.
2. This extended equation is obtained by changing the classical derivative d/dt by a new operator acting on non-differentiable functions and recovering d/dt over differentiable ones.
3. If the initial equation has a specific structure, like being solution of a variational problem, one must recover the analogous of this structure in the non-differentiable context.

The last assumption is related to Nottale's interpretation of the relativity principle. The non-differentiable embedding gives a mathematical meaning to 1 and 2. For 3, we mainly focus on the non-differentiable embedding of the Euler–Lagrange equation which governs most of the dynamical behavior of physical systems in classical mechanics and physics [2]. The Euler–Lagrange equation comes from a variational principle called the least-action principle which is one of the fundamental principle of physics. The least-action principle is based on a functional called the action functional, also called the Lagrangian functional, which is completely determined by a scalar function called the Lagrangian. The mathematical tool to study these functionals is the classical calculus of variations.

Using the non-differentiable embedding we obtain a natural non-differentiable analogous of this equation. However, this result is by itself not sufficient. Indeed, as we have a non-differentiable analogue of Euler–Lagrange equation, we are led to the following problems:

1. Develop a calculus of variations on non-differentiable functionals.
2. State the corresponding *non-differentiable least-action principle*, in particular explicit the associated *non-differentiable Euler–Lagrange equation* denoted by NDEL.
3. Compare the result with the non-differentiable embedded Euler–Lagrange equation Emb(EL).

Following our previous work [5], we develop in this paper a non-differentiable calculus of variations and we obtain the corresponding non-differentiable least-action principle. We prove that the non-differentiable embedding is coherent, *i.e.* that we have NDEL = Emb(*EL*). As a consequence, our construction takes into account the underlying variational nature of the Euler–Lagrange equation in the non-differentiable case.

We then provide two applications of this formalism. First, we extend the classical characteristics method for PDEs to the non-differentiable setting. As a consequence, we can consider PDEs of mixed type like the Navier–Stokes equation. We prove that the non-differentiable characteristic of the Navier–Stokes equation are non-differentiable extremals of an explicit Lagrangian functional. The non-differentiability is related to the viscosity coefficient. Second, following [5], we obtain a non-differentiable Lagrangian structure for the Schrödinger equation in arbitrary dimension.

The Schrödinger equation has been discussed in [5] and [4]. The article [4] contains a heuristic version of the idea of embedding for differential operators called *scale quantization procedure* which is used to obtain Eq. (43) with quantum calculus. However, this derivation of the Schrödinger equation can be seen as a formal substitution of the classical derivative by a quantum derivative without an underlying "physical" principle giving the equation. In [5] we try to bypass this problem by providing a non-conventional variational formulation of Eq. (43) using non-differentiable functionals and quantum calculus. However, in this case the form of the functional is given a priori and justified from the physical side by the *scale relativity principle* (see Section 5.2 in [5, p. 63]). The non-differentiable embedding point of view that we introduce in this paper can be used to unify the two results in [4] and [5] by providing both Eq. (43) and the form of the functional (27) in the quantum calculus setting from the Newton's Lagrangian (42) and provides a strong connection between the classical Newton's equation and the Schrödinger equation. The Schrödinger equation can be seen as the Newton's equation over a non-differentiable space. The passage from Newton's to Schrödinger preserves an essential feature of classical mechanics which is the fact to be a critical point of a functional, *i.e.* the underlying Lagrangian structure.

Moreover, the formalism that we develop in this paper is different from the one in [5] and [4]. In particular, we have changed the definition of the operator acting on non-differentiable functions. The reason is that quantum calculus let a free parameter (ϵ in [5]) which is present in all the computations. In particular, the derivation of the Schrödinger equation in [5] is done assuming that a quantity depending on ϵ is equal to a constant (see Eq. (36) in [5, p. 60]). This condition is certainly difficult to satisfy and the interpretation of the free parameter in this case is not clear. The new operator $\Box/\Box t$ that we introduce in this paper bypasses this problem by associating to a non-differentiable function a quantity which is free of extra parameters and which reduces to the classical derivative for differentiable functions.

The paper is organized as follows:

In Section 1, we first give some notations, and discuss about the irreversibility. Then, we define the non-differentiable embedding. Section 2 is devoted to the non-differentiable embedding of Lagrangian, while Section 3 concerns the non-differentiable embedding of Hamiltonian systems. In both cases, we get a coherence principle for the Lagrangian system, meaning that the following diagram is commutative.

$$\begin{tikzcd}
\text{Lagrangian} \arrow[r, "\text{N.D. Emb}"] \arrow[d, "\text{L.A.P.}"'] & \text{N.D. Lagrangian} \arrow[d, "\text{N.D.L.A.P.}"] \\
\text{Euler–Lagrange equation} \arrow[r, "\text{N.D. Emb}"] & \text{N.D. Euler–Lagrange equation}
\end{tikzcd}$$

In Section 4, we discuss applications to PDEs and in particular the Schrödinger and the Navier–Stokes equations.

1. About non-differentiable embedding

1.1. Introduction

Classical equations of mechanics or physics are written using differential or partial differential equations. These equations are derived from experimental data measuring for example the successive positions x_i, $i = 1, \dots, n$, of a particle at times t_i, $i = 1, \dots, n$ respectively. Assuming that the particle moves continuously[1] along a path $x(t)$, $t \in \mathbb{R}$, we denote $x(t_i) := x_i$. Usually one also computes the mean velocity v_i of the particle during the interval of time $\Delta t_i = t_i - t_{i-1}$ for $i = 2, \dots, n$. Using all these quantities we are able to construct a differential equation of the form

$$\frac{d_\epsilon^- x}{dt}(t) = v\big(x(t), t, \epsilon\big),$$

where v, the velocity, is determined by the experience and d_ϵ^- stands for the ϵ-mean left derivative, *i.e.*

$$\frac{d_\epsilon^- x}{dt}(a) := \frac{x(a) - x(a-\epsilon)}{\epsilon}.$$

Assuming that the quantity d_ϵ^- has a limit when ϵ goes to zero, we obtain a *backward differential equation*

[1] This assumption is of course far from being trivial in particular when dealing with quantum mechanics and is called the *assumption of continuity* by E. Schrödinger [16].

$$\frac{d^- x}{dt}(t) = F\big(x(t), t\big),$$

where d^- is the left derivative of x, *i.e.*

$$\frac{d^- x}{dt}(a) := \lim_{t \to a^-} \frac{x(a) - x(t)}{a - t}.$$

In general one cannot produce the corresponding *forward differential equation*, because of the existence of an *arrow of time*. We only have access to a set of informations concerning the past of the particle. In order to use the data to construct the forward differential equation one makes a strong assumption namely the *reversibility* assumption, precisely it means that the arrow of time does not come into play. In that case, the forward differential equation can be deduced from the backward one and we obtain an ordinary differential equation

$$\frac{dx}{dt}(t) = F(x, t),$$

which encodes this assumption.

Rejecting the reversibility assumption we must deal with a set of two differential equations, the forward and backward one, that describes completely the dynamical behavior of the particle, *i.e.*

$$\begin{cases} \dfrac{d^- x}{dt}(t) = F^-(x, t), \\ \dfrac{d^+ x}{dt}(t) = F^+(x, t) \end{cases} \tag{1}$$

which leaves open the question of reversibility/irreversibility of the underlying process.

One can introduce a new operator taking into account $d^\pm$ using complex numbers. Indeed, we define a complex-valued operator denoted by D_μ and defined by

$$D_\mu := \frac{d^+ + d^-}{2} + i\mu \frac{d^+ - d^-}{2},$$

where $\mu \in \mathbb{C}$ is only of use to recover special cases of interest, *i.e.* for $\mu = i$, $D_i = d^-$ and for $\mu = -i$, $D_{-i} = d^+$.

Remark 1. We can go further by taking two independent variables x^+ and x^- for the forward and backward time evolution of the physical system. This will be explored in a forthcoming paper. In the case we need to *weight* the dynamical information coming from the past or the future, we could use the *fractional* version of this construction. We refer to [7] for more details.

Our discussion about reversibility/irreversibility justifies the introduction of the left and right derivatives. In order to cover non-differentiable curves we consider a weaker notion of derivatives given by the quantum calculus [13].

1.2. Reminder about the quantum calculus

In this subsection, we recall the definition as well as the basic properties of the quantum calculus and the ϵ-scale derivatives introduced in [4] and [5] (see also [13]).

1.2.1. Notations

Let $d \in \mathbb{N}$ be a fixed integer, I an open set in $\mathbb{R}$, and $a, b \in \mathbb{R}$, $a < b$, such that $[a, b] \subset I$, be given in the whole paper. We denote by $\mathcal{F}(I, \mathbb{R}^d)$ the set of functions $x : I \to \mathbb{R}^d$ from I to $\mathbb{R}^d$, and $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) the subset of $\mathcal{F}(I, \mathbb{R}^d)$ (respectively $\mathcal{F}(I, \mathbb{C}^d)$) which are continuous. Let $n \in \mathbb{N}$, we denote by $\mathcal{C}^n(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^n(I, \mathbb{C}^d)$) the set of functions in $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) which are differentiable up to order n.

Definition 1 *(Hölderian functions).* Let $w \in \mathcal{C}^0(I, \mathbb{R}^d)$. Let $t \in I$.

1. w is Hölder of Hölder exponent α, $0 < \alpha < 1$, at point t if
$$\exists c > 0,\ \exists \eta > 0 \quad \text{s.t.} \quad \forall t' \in I \quad \big|t - t'\big| \leqslant \eta \quad \Rightarrow \quad \big\|w(t) - w\big(t'\big)\big\| \leqslant c\big|t - t'\big|^\alpha,$$
where $\|\cdot\|$ is a norm on $\mathbb{R}^d$.
2. w is α-Hölder and inverse Hölder with $0 < \alpha < 1$, at point t if
$$\exists c, C \in \mathbb{R}^{+*},\ c < C,\ \exists \eta > 0 \quad \text{s.t.} \quad \forall t' \in I \big|t - t'\big| \leqslant \eta$$
$$c\big|t - t'\big|^\alpha \leqslant \big\|w(t) - w\big(t'\big)\big\| \leqslant C\big|t - t'\big|^\alpha.$$

A complex-valued function is α-Hölder if its real and imaginary parts are α-Hölder. We denote by $H^\alpha(I,\mathbb{R}^d)$ the set of continuous functions α-Hölder and by $\mathbb{H}^\alpha(I,\mathbb{R}^d)$ the set of continuous functions α-Hölder and α-inverse Hölder. For explicit examples of α-Hölder and α-inverse Hölder functions we refer to [17, p. 168] in particular the Knopp or Takagi function.

1.2.2. Quantum calculus

For a general continuous function, we cannot define the derivative at a given point. However, for all $\epsilon > 0$, we have access to the left and right quantum derivatives, which are the *discrete* versions of the left and right derivatives.

Definition 2. Let $x \in \mathcal{C}^0(I,\mathbb{R}^d)$. For all $\epsilon > 0$, we call ϵ-left and right quantum derivatives the quantities $d_\epsilon^\sigma x$ defined for any $t \in I$ by

$$d_\epsilon^\sigma x(t) := \sigma\frac{x(t+\sigma\epsilon) - x(t)}{\epsilon}, \quad \sigma = \pm.$$

The ϵ-left and right quantum derivatives of a continuous function correspond to the classical derivatives of the left and right ϵ-mean function defined by

$$x_\epsilon^\sigma(t) := \frac{\sigma}{\epsilon}\int_t^{t+\sigma\epsilon} x(s)\,ds, \quad \sigma = \pm.$$

Then, d_ϵ^σ can be interpreted as the left and right derivatives at a given scale ϵ.

Using ϵ-left and right derivatives, we can define the quantum derivative, $\frac{\Box_\epsilon x}{\Box t}$, which generalizes the classical derivative.

Definition 3. Let $x \in \mathcal{C}^0(I,\mathbb{R}^d)$. For any $\epsilon > 0$, the ϵ-scale derivative of x is the quantity denoted by $\frac{\Box_\epsilon}{\Box t} : \mathcal{C}^0(I,\mathbb{R}^d) \to \mathcal{C}^0(I,\mathbb{C}^d)$, and defined at point t by

$$\frac{\Box_\epsilon x}{\Box t}(t) := \frac{1}{2}\left[\left(d_\epsilon^+ x(t) + d_\epsilon^- x(t)\right) + i\mu\left(d_\epsilon^+ x(t) - d_\epsilon^- x(t)\right)\right],$$

where $\mu \in \{1,-1,0,i,-i\}$.

If x is differentiable, taking the limit of the ϵ-scale derivative when ϵ goes to zero, leads to $\frac{dx}{dt}$ the classical derivative of x. We will frequently denote $\Box_\epsilon x$ for $\frac{\Box_\epsilon x}{\Box t}$. Moreover, we do not write the dependence on $\Box_\epsilon$ on μ. Let us notice that for $\mu = i$ we obtain d_ϵ^- and for $\mu = -i$ we get d_ϵ^+, which allow us to recover the backward and forward derivatives.

We also need to extend the scale derivative to complex-valued functions in order to be able to compute *composition* of $\Box_\epsilon$. Indeed, if x is a real-valued function $\Box_\epsilon x$ is a complex-valued function, and it is not clear what is the correct definition of $\Box_\epsilon(\Box_\epsilon x)$. Then, we choose to define $\Box_\epsilon$ over complex-valued functions as follows:

Definition 4. Let $x \in \mathcal{C}^0(I,\mathbb{C}^d)$ be a continuous complex-valued function. For all $\epsilon > 0$, the ϵ-scale derivative of x, denoted by $\frac{\Box_\epsilon x}{\Box t}$ is defined by

$$\frac{\Box_\epsilon x}{\Box t} := \frac{\Box_\epsilon \operatorname{Re}(x)}{\Box t} + i\frac{\Box_\epsilon \operatorname{Im}(x)}{\Box t}, \tag{2}$$

where $\operatorname{Re}(x)$ and $\operatorname{Im}(x)$ are the real and imaginary parts of x.

This extension of the quantum derivative in order to cover complex-valued functions is far from being trivial. Indeed, it mixes complex terms in a complex operator. From an algebraic point of view it means the operator $\Box_\epsilon$ has to be $\mathbb{C}$-linear (see also [9]).

1.2.3. Box derivative

The previous operator depends on ϵ which is a free parameter related to the smoothing order of the function. This induces difficulties in applications to physics when one is interested in particular equations which do not depend on an extra parameter, like the Schrödinger equation. In the following we introduce a procedure to extract an information independent of ϵ but related to the mean behavior of the function.

Let $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ be a sub-vectorial space of $\mathcal{C}^0(I\times]0,1],\mathbb{R}^d)$ such that for any function $f \in \mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ the limit $\lim_{\epsilon\to 0} f(t,\epsilon)$ exists for any $t \in I$. We denote by E the complementary space of $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ in $\mathcal{C}^0(I\times]0,1],\mathbb{R}^d)$ and by π the projection onto $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ by

$$\pi : \mathcal{C}^0_{conv}\left(I\times]0,1],\mathbb{R}^d\right)\oplus E \to \mathcal{C}^0_{conv}\left(I\times]0,1],\mathbb{R}^d\right) \tag{3}$$

$$f_{conv} + f_E \mapsto f_{conv}. \tag{4}$$

We can then define the operator $\langle\,.\,\rangle$ by

$$\langle\,.\,\rangle : \mathcal{C}^0\big(I\times\,]0,1],\mathbb{R}^d\big) \to \mathcal{F}\big(I,\mathbb{R}^d\big)$$
$$f \mapsto \langle f\rangle : t \mapsto \lim_{\epsilon\to 0} \pi(f)(t,\epsilon).$$

Definition 5. Let us introduce the new operator $\frac{\Box}{\Box t}$ (without ϵ) on the space $\mathcal{C}^0(I,\mathbb{R}^d)$ by:

$$\frac{\Box x}{\Box t} := \left\langle \frac{\Box_\epsilon x}{\Box t}\right\rangle. \tag{5}$$

Remark 2. This operator depends on the choice of the supplementary space E. However this dependence does not change the form of the formula we obtain in the following subsections. We give an interpretation in Section 1.4 of this dependency under a particular functional space and we then discuss the physical meaning of this dependence.

The operator $\frac{\Box}{\Box t}$ acts on complex-valued functions by $\mathbb{C}$-linearity.

For a differentiable function $x \in \mathcal{C}^1(I,\mathbb{R}^d)$, $\frac{\Box x}{\Box t} = \frac{dx}{dt}$, which is the classical derivative. More generally if $\frac{\Box^k x}{\Box t^k}$ denotes $\frac{\Box^k x}{\Box t^k} := \frac{\Box}{\Box t}\circ\cdots\circ\frac{\Box}{\Box t}x$ and $x\in\mathcal{C}^k(I,\mathbb{R}^d)$, $k\in\mathbb{N}$, then $\frac{\Box^k x}{\Box t^k} = \frac{d^k x}{dt^k}$.

Remark 3. In [5] we introduce a bracket $[\,.\,]_\epsilon$ which keeps the divergent information. It was defined as:

$$\big[\Box_\epsilon x(t)\big]_\epsilon := \Box x + x_E(t,\epsilon),$$

where x_E denotes the projection of x on E. As a consequence, we obtain more complicated quantities and in particular we keep a dependence on ϵ.

1.3. Properties of the Box derivative

In the following, we derive the counter part of the classical Leibniz formula, composition rule formula and fundamental theorem of differential calculus for the Box derivative. The form of these formulas does not depend on the choice made for the supplementary space E in Definition 5.

1.3.1. Non-differentiable Leibniz formula

Lemma 1. *Let $f\in H^\alpha(I,\mathbb{R}^d)$ and $g\in H^\beta(I,\mathbb{R}^d)$, with $\alpha+\beta>1$,*

$$\frac{\Box}{\Box t}(f\cdot g) = \frac{\Box f}{\Box t}\cdot g + f\cdot\frac{\Box g}{\Box t}. \tag{6}$$

Proof. Let us first consider real-valued functions $f\in H^\alpha(I,\mathbb{R})$ and $g\in H^\beta(I,\mathbb{R})$, and start with d_ϵ^+, then

$$d_\epsilon^+(fg) = \big(d_\epsilon^+ f\big)g + f\big(d_\epsilon^+ g\big) + \epsilon\big(d_\epsilon^+ f\big)\big(d_\epsilon^+ g\big).$$

Since f and g are respectively α- and β-Hölder, we have $|d_\epsilon^+ f| \leqslant c_f\epsilon^{\alpha-1}$ and $|d_\epsilon^+ g|\leqslant c_g\epsilon^{\beta-1}$, then $|\epsilon d_\epsilon^+ f d_\epsilon^+ g| \leqslant c_f c_g \epsilon^{\alpha+\beta-1}$. This quantity converges to 0, when ϵ goes to 0, since $\alpha+\beta>1$, so that $\langle \epsilon d_\epsilon^+(fg)\rangle = 0$. The same holds for d_ϵ^-, and we obtain

$$\big\langle d_\epsilon^\sigma(fg)\big\rangle = \big\langle d_\epsilon^\sigma f\big\rangle g + f\big\langle d_\epsilon^\sigma g\big\rangle.$$

By linearity, we get $\frac{\Box}{\Box t}(fg) = \frac{\Box f}{\Box t}.g + f.\frac{\Box g}{\Box t}$ for real-valued functions. The generalization to vector-valued functions $f\in H^\alpha(I,\mathbb{R}^d)$ and $g\in H^\beta(I,\mathbb{R}^d)$ is straightforward. $\Box$

1.3.2. Fundamental result for the Box derivative

In the following subsection, we assume that a supplementary space E is fixed and we keep the notations introduced in (3).

Definition 6. We denote by $\mathcal{C}^1_\Box(I,\mathbb{R})$ the set of continuous functions $f\in\mathcal{C}^0(I,\mathbb{R})$ such that $\frac{\Box f}{\Box t}\in\mathcal{C}^0(I,\mathbb{R})$.

Lemma 2. *Let $f \in \mathcal{C}^1_\Box(I,\mathbb{R})$ be such that*

$$\lim_{\epsilon\to 0}\int_a^b (\Box_\epsilon f)_E\,dt = 0 \tag{7}$$

then

$$\int_a^b \frac{\Box f(t)}{\Box t}\,dt = f(b)-f(a). \tag{8}$$

Proof. With the definition of d_ϵ^σ, it is easy to check that $\lim_{\epsilon\to 0}\int_a^b d_\epsilon^\sigma f(t)\,dt = f(b)-f(a)$, $\sigma=\pm$. We deduce that

$$\lim_{\epsilon\to 0}\int_a^b \Box_\epsilon f(t)\,dt = f(b)-f(a).$$

Moreover,

$$\int_a^b \frac{\Box f(t)}{\Box t}\,dt = \lim_{\epsilon\to 0}\int_a^b \pi\big(\Box_\epsilon f(t)\big)\,dt.$$

By $\Box_\epsilon f(t) - \pi\big(\Box_\epsilon f(t)\big) = \big(\Box_\epsilon f(t)\big)_E$ and the condition (7), we deduce that

$$\lim_{\epsilon\to 0}\int_a^b \big(\Box_\epsilon f(t) - \pi\big[\Box_\epsilon f(t)\big]\big)\,dt = \lim_{\epsilon\to 0}\int_a^b \big(\Box_\epsilon f(t)\big)_E = 0.$$

We deduce the result. $\Box$

1.3.3. Box composition rule

For $x\in\mathcal{C}^1(I,\mathbb{R})$ and $f\in\mathcal{C}^1(I,\mathbb{R})$, the classical differential calculus gives

$$\frac{df\big(x(t),t\big)}{dt} = \frac{\partial f}{\partial t}\big(x(t),t\big) + \frac{\partial f}{\partial x}\big(x(t),t\big)\cdot x'(t).$$

The analogous of the derivative of a composed function for the quantum derivative is given by the following theorem:

Theorem 1. *Let f be a $C^2(\mathbb{R}^d\times I,\mathbb{R})$ function and $x\in H^\alpha(I,\mathbb{R}^d)$, $\frac{1}{2}\leqslant\alpha<1$, then we have*

$$\frac{\Box f(x(t),t)}{\Box t} = \nabla_x f\big(x(t),t\big)\cdot\frac{\Box x}{\Box t}(t) + \frac{\partial f}{\partial t}\big(x(t),t\big) + \sum_{k=1}^d\sum_{l=1}^d \frac{1}{2}\frac{\partial^2 f}{\partial x_k\partial x_l}\big(x(t),t\big)a_{k,l}\big(x(t)\big) \tag{9}$$

where

$$a_{k,l}\big(x(t)\big) := \left\langle \frac{\epsilon}{2}\Big(\big(d_\epsilon^+ x_k(t)\big)\big(d_\epsilon^+ x_l(t)\big)(1+i\mu) - \big(d_\epsilon^- x_k(t)\big)\big(d_\epsilon^- x_l(t)\big)(1-i\mu)\Big)\right\rangle \tag{10}$$

where i is the complex number.

Note that $a_{k,l}(x)=0$ if $x\in\mathcal{C}^1(I,\mathbb{R}^d)$.

Proof of Theorem 1. Let us first consider d_ϵ^+:

$$d_\epsilon^+\big(f\big(x(t),t\big)\big) = \frac{f(x(t+\epsilon),t+\epsilon) - f(x(t),t)}{\epsilon}.$$

Moreover

$$f\big(x(t+\epsilon),t+\epsilon\big) = f\big(\epsilon d_\epsilon^+\big(x(t)\big) + x(t), t+\epsilon\big).$$

As x is α-Hölder, $\|d_\epsilon^+(x)\| \leqslant c\epsilon^{\alpha-1}$, and $\epsilon\|d_\epsilon^+(x)\| \leqslant c\epsilon^\alpha$, $\lim_{\epsilon\to 0}\epsilon\|d_\epsilon^+(x)\| = 0$. Since f is of class $\mathcal{C}^2$, we can consider its expansion:

$$\begin{aligned}
f\big(x(t+\epsilon), t+\epsilon\big) = {} & f\big(x(t),t\big) + \sum_{k=1}^{d} \frac{\partial f}{\partial x_k}\big(x(t),t\big)\epsilon d_\epsilon^+\big(x_k(t)\big) + \frac{\partial f}{\partial t}\big(x(t),t\big)\epsilon \\
& + \sum_{k=1}^{d}\sum_{l=1}^{d} \frac{1}{2!}\frac{\partial^2 f}{\partial x_k \partial x_l}\big(x(t),t\big)\epsilon d_\epsilon^+\big(x_k(t)\big)\epsilon d_\epsilon^+\big(x_l(t)\big) \\
& + \frac{1}{2!}\frac{\partial^2 f}{\partial^2 t}\big(x(t),t\big)\epsilon^2 + \sum_{k=1}^{d}\frac{\partial^2 f}{\partial x_k \partial t}\big(x(t),t\big)\epsilon d_\epsilon^+\big(x_k(t)\big)\epsilon \\
& + o\Big(\big\|\big(\epsilon d_\epsilon^+\big(x_1(t)\big), \ldots, \epsilon d_\epsilon^+\big(x_d(t)\big), \epsilon\big)\big\|^2\Big).
\end{aligned}$$

Since $\|\epsilon d_\epsilon^+ x_k(t)\| \leqslant c_k\epsilon^\alpha$, for any $k = 1, \ldots, d$, and $\frac{1}{2} \leqslant \alpha < 1$, we get

$$o\Big(\big\|\big(\epsilon d_\epsilon^+\big(x_1(t)\big), \ldots, \epsilon d_\epsilon^+\big(x_d(t)\big), \epsilon\big)\big\|^2\Big) \leqslant \sum_{k=1}^{d} c_k^2\epsilon^{2\alpha} + \epsilon^2 < c\epsilon^{2\alpha}.$$

We obtain the following formula for d_ϵ^+:

$$\begin{aligned}
d_\epsilon^+\big(f\big(x(t),t\big)\big) = {} & \sum_{k=1}^{d}\frac{\partial f}{\partial x_k}\big(x(t),t\big)\, d_\epsilon^+\big(x_k(t)\big) + \frac{\partial f}{\partial t}\big(x(t),t\big) \\
& + \sum_{k=1}^{d}\sum_{l=1}^{d}\frac{1}{2!}\frac{\partial^2 f}{\partial x_k \partial x_l}\big(x(t),t\big)\epsilon\, d_\epsilon^+\big(x_k(t)\big)\, d_\epsilon^+\big(x_l(t)\big) + o\big(\epsilon^{2\alpha-1}\big).
\end{aligned} \tag{11}$$

Doing the same calculation for $d_\epsilon^- f$ gives:

$$\begin{aligned}
d_\epsilon^-\big(f\big(x(t),t\big)\big) = {} & \sum_{k=1}^{d}\frac{\partial f}{\partial x_k}\big(x(t),t\big)\, d_\epsilon^-\big(x_k(t)\big) + \frac{\partial f}{\partial t}\big(x(t),t\big) \\
& - \sum_{k=1}^{d}\sum_{l=1}^{d}\frac{1}{2!}\frac{\partial^2 f}{\partial x_k \partial x_l}\big(x(t),t\big)\epsilon\, d_\epsilon^-\big(x_k(t)\big)\, d_\epsilon^-\big(x_l(t)\big) + o\big(\epsilon^{2\alpha-1}\big).
\end{aligned} \tag{12}$$

Finally, combining (11) and (12) leads to the result. $\square$

The previous result can be generalized for arbitrary values of $0 < \alpha < 1$ following the same computations but taking into account that for $x \in H^\alpha$ with $1/n \leqslant \alpha < 1/(n-1)$ we choose $f \in C^n$. In the applications that we discuss in Section 4, we only deal with the case $1/2 \leqslant \alpha < 1$.

1.4. $C_{W,\lambda}^{k,\alpha}$ functions

In this subsection we introduce a functional set which will be important for applications. The idea is to describe functions for which the $\square$-derivative can be easily computed and interpreted.

1.4.1. Standard irregular functions

We first define a special class of functions called **standard irregular functions**. These functions contain the minimal irregular behavior we are interested in for applications. In the following, we assume that we have fixed a definition for the $\square$-derivative, meaning that a special choice for the supplementary space E in Section 1.2.3 has been made.

Definition 7. Let $0 < \alpha < 1$, then a standard irregular function of order α is a function $W_\alpha \in \mathbb{H}^\alpha$ such that $\square W_\alpha = 0$.

As $W_\alpha \in \mathbb{H}^\alpha$ the quantity $\square_\epsilon W$ diverges to ∞. As a consequence, we can choose the supplementary space E_α such that $\square_\epsilon W_\alpha \in E_\alpha$ and we obtain $\square W_\alpha = 0$. In other words, for a fixed irregular function W we can find an adapted supplementary space E such that $\square W = 0$.

We can now introduce the functional set which we consider for applications.

Definition 8. Let $0 < \alpha < 1$ and W_α be a standard irregular function of order α. We denote by $\mathcal{C}^{k,\alpha}_{W_\alpha}(I, \mathbb{C}^d)$ the subset of $\mathcal{C}^0(I, \mathbb{C}^d)$ defined by

$$\mathcal{C}^{k,\alpha}_{W_\alpha} = \{x := u + \lambda.W_\alpha,\ u \in \mathcal{C}^k(I, \mathbb{C}^d),\ \lambda \in \mathbb{C}\}. \tag{13}$$

The previous set is a sum of a regular differentiable part and a standard irregular function of order α. From a mathematical point of view, this functional set is close to paths obtained for diffusion processes in stochastic calculus [12].

Remark 4. In physics many problems involve at least two different scales. For example, when one considers classical mechanics, the structure of space–time is smooth and a trajectory on it is usually smooth. In microphysics however, the space–time loses its smoothness and becomes irregular. Nottale [14] assumes a fractal structure for it. As a consequence a trajectory in microphysic can be understood as a regular part which is controlled by classical law of physics and an irregular part which is important at very small scale which induces irregularities and non-differentiability.

For many applications, the irregular part is not free or does not need to be free. In particular, we consider the subset of $\mathcal{C}^{k,\alpha}_{W_\alpha}(I, \mathbb{C}^d)$ denoted by $\mathcal{C}^{k,\alpha}_{\lambda, W_\alpha}(I, \mathbb{C}^d)$ and defined for each $\lambda \in \mathbb{C}$ by

$$\mathcal{C}^{k,\alpha}_{\lambda, W_\alpha} = \{x := u + \lambda.W_\alpha,\ u \in \mathcal{C}^k(I, \mathbb{C}^d)\}. \tag{14}$$

1.4.2. The special case: $1/2 \leqslant \alpha < 1$

Formula (9) can be specialized over a set $\mathcal{C}^{k,\alpha}_{W_\alpha}$. We first prove the following lemma:

Lemma 3. *Let $x \in \mathcal{C}^{1,\alpha}_{W_\alpha}(I, \mathbb{R}^d)$, then $x = u + \lambda W_\alpha$, $u \in C^1$, $\lambda \in \mathbb{C}$, and*

$$a_{k,l}(x) = \lambda^2 a_{k,l}(W_\alpha).$$

Proof. A simple computation gives

$$a_{k,l}(x) = a_{k,l}(u) + \lambda^2 a_{k,l}(W_\alpha) + \lambda M_{k,l}(u, W_\alpha),$$

where the mixed term $M_{k,l}(u, W_\alpha)$ is given by

$$M_{k,l}(u, W_\alpha) := \left\langle \frac{\epsilon}{2}\Big(\big(d^+_\epsilon W_{\alpha,k}(t)\big)\big(d^+_\epsilon u_l(t)\big)(1 + i\mu) - \big(d^-_\epsilon W_{\alpha,k}(t)\big)\big(d^-_\epsilon w_l(t)\big)(1 - i\mu)\Big)\right\rangle.$$

As $u \in C^1$, we have $a_{k,l}(u) = 0$. Moreover, as $\epsilon d^\sigma_\epsilon W_{\alpha,j}(t)$ goes to zero when ϵ goes to zero and $d^\sigma_\epsilon u_j(t)$ converges toward du_j/dt, we deduce that the operator $\langle\cdot\rangle$ sends all the terms in $M_{k,l}(W_\alpha)$ to zero. $\square$

The following lemma gives a particular importance to the value $\alpha = 1/2$:

Lemma 4. *Let $0 < \alpha < 1$ and W_α be a standard irregular function, then the quantities $a_{k,l}(W_\alpha) = 0$ for all $k, l = 1, \dots, d$ except for $\alpha = 1/2$.*

We then introduce the two following notion:

Definition 9. A canonical (resp. anti-canonical) irregular function is a standard irregular function denoted by $\mathbb{W}$ (resp. $\mathbf{W}$) of order $1/2$ such that

$$a_{k,l}(\mathbb{W}) = \delta_{k,l} \quad \big(\text{resp. } a_{k,l}(\mathbf{W}) = -\delta_{k,l}\big). \tag{15}$$

The $\square$ composition rule formula takes an interesting form on $\mathcal{C}^{k,1/2}_{\mathbb{W}}$ (resp. $\mathcal{C}^{k,1/2}_{\mathbf{W}}$). Indeed, we have:

Corollary 1. *Let f be a $\mathcal{C}^2(\mathbb{R}^d \times I, \mathbb{R})$ function.*

- *If $x = (x_1, \dots, x_d) \in \mathcal{C}^{1,1/2}_{\mathbb{W}}(I, \mathbb{R}^d)$ written as $x := u + \lambda.\mathbb{W}$ where $u = (u_1, \dots, u_d) \in \mathcal{C}^1(I, \mathbb{R}^d)$ and $\lambda \in \mathbb{C}$ then the following formula holds*

$$\frac{\square f(x(t), t)}{\square t} = \nabla_x f\big(x(t), t\big) \cdot \frac{du}{dt}(t) + \frac{\partial f}{\partial t}\big(x(t), t\big) + \frac{\lambda^2}{2}\Delta f\big(x(t), t\big). \tag{16}$$

– *If* $x=(x_1,\dots,x_d)\in C_{\mathbf{W}}^{1,1/2}(I,\mathbb{R}^d)$ *written as* $x:=u+\lambda\mathbf{W}$ *where* $u=(u_1,\dots,u_d)\in\mathcal{C}^1(I,\mathbb{R}^d)$ *and* $\lambda\in\mathbb{C}$ *then the following formula holds*

$$\frac{\Box f(x(t),t)}{\Box t}=\nabla_x f\big(x(t),t\big)\cdot\frac{du}{dt}(t)+\frac{\partial f}{\partial t}\big(x(t),t\big)-\frac{\lambda^2}{2}\Delta f\big(x(t),t\big). \tag{17}$$

The irregular part of x gives rise to the Laplacian part of the formula.

1.5. Non-differentiable embedding

This subsection follows the strategy of the stochastic and fractional embeddings of differential operators (see [7] and [9]).

1.5.1. Non-differentiable embedding of differential operators

Let $f:I\times\mathbb{C}^d\to\mathbb{C}$ be a function continuous, real-valued on real arguments. We denote by F the corresponding operator acting on functions x and defined by

$$\mathrm{F}:\begin{array}{l}\mathcal{C}^0(I,\mathbb{C}^d)\to\mathcal{C}^0(I,\mathbb{C})\\ \qquad\qquad x\mapsto f\big(\bullet,x(\bullet)\big)\end{array}$$

where $f(\bullet,x(\bullet))$ is the function defined by

$$f\big(\bullet,x(\bullet)\big):\begin{array}{l}I\to\mathbb{C},\\ t\mapsto f\big(t,x(t)\big).\end{array}$$

Let $\mathbf{f}=\{f_i\}_{i=0,\dots,n}$ (resp. $\mathbf{g}=\{g_i\}_{i=0,\dots,n}$) be a finite family of functions of class $\mathcal{C}^n$, $f_i:I\times\mathbb{C}^d\to\mathbb{C}$ (resp. $g_i:I\times\mathbb{C}^d\to\mathbb{C}$), and F_i (resp. G_i), $i=0,\dots,n$ the corresponding family of operators.

Definition 10. We denote by $\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}$ the differential operator acting on $\mathcal{C}^n(I,\mathbb{C}^d)$ defined by

$$\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}=\sum_{i=0}^{n}\mathrm{F}_i\cdot\left(\frac{d^i}{dt^i}\circ\mathrm{G}_i\right), \tag{18}$$

where $\cdot$ is the standard product of operators, *i.e.* if A and B are two operators, we denote by $A\cdot B$ the operator defined by $(A\cdot B)(x)=A(x)B(x)$ and $\circ$ the usual composition, *i.e.* $(A\circ B)(x)=A(B(x))$, with the convention that $(\frac{d}{dt})^0=\mathrm{Id}$, where Id denotes the identity mapping on $\mathbb{C}$.

Definition 11 *(Non-differentiable embedding of operators).* The non-differentiable embedding of $\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}$ written as (18), denoted by $\mathrm{Emb}_{\Box}(\mathrm{O}_{\mathbf{f}}^{\mathbf{g}})$ is the operator

$$\mathrm{Emb}_{\Box}\big(\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}\big)=\sum_{i=0}^{n}F_i\cdot\left(\frac{\Box^i}{\Box t^i}\circ G_i\right). \tag{19}$$

Note that the embedding procedure acts on operators of *a given form* and not on operators like abstract data, *i.e.* this is not a mapping on the set of operators.

1.5.2. Non-differentiable embedding of differential equations

Let $k\in\mathbb{N}$ be a fixed integer. Let $\mathbf{f}=\{f_i\}_{i=0,\dots,n}$ and $\mathbf{g}=\{g_i\}_{i=0,\dots,n}$ be finite families of functions of class $\mathcal{C}^n$, $f_i:\mathbb{R}\times\mathbb{C}^{kd}\to\mathbb{C}$ and $g_i:\mathbb{R}\times\mathbb{C}^{kd}\to\mathbb{C}$ respectively, and F_i, G_i, $i=0,\dots,n$ the corresponding families of operators. We denote by $\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}$ the operator acting on $\mathcal{C}^n(I,\mathbb{R})\times\mathcal{C}^n(I,\mathbb{C}^d)\times\cdots\times\mathcal{C}^{k+n}(I,\mathbb{C}^d)$ defined by

$$\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}=\sum_{i=0}^{n}F_i\cdot\left(\frac{d^i}{dt^i}\circ G_i\right).$$

Definition 12. Let the ordinary differential equation associated to $\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}$ be defined by

$$\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}\left(\bullet,x(\bullet),\frac{dx}{dt}(\bullet),\dots,\frac{d^kx}{dt^k}(\bullet)\right)=0,\quad\text{for any } x\in\mathcal{C}^{k+n}(I,\mathbb{C}). \tag{20}$$

We then define the non-differentiable embedding of Eq. (20) as follows:

Definition 13. The non-differentiable embedding of Eq. (20) is defined by

$$\mathrm{Emb}_{\Box}\big(\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}\big)\left(\bullet, x(\bullet), \frac{\Box x}{\Box t}(\bullet), \ldots, \frac{\Box^k x}{\Box t^k}(\bullet)\right) = 0, \quad x \in \mathcal{C}_{\Box}^{k+n}\big(I, \mathbb{C}^d\big), \tag{21}$$

where $\mathcal{C}_{\Box}^{k+n}(I, \mathbb{C}^d)$ denotes the set of functions $x \in \mathcal{C}^0(I, \mathbb{C}^d)$ such that $\Box^i x \in \mathcal{C}^0(I, \mathbb{C}^d)$ for $i = 1, \ldots, k+n$.

Note that as long as the form of the operator is fixed the non-differentiable embedding procedure associates a *unique* equation.

In the following section, we explicit the non-differentiable embedding of a particular class of differential equations called Lagrangian systems.

2. Non-differentiable embedding of Lagrangian systems and coherence principle

2.1. Non-differentiable embedding for Euler–Lagrange equation

2.1.1. Reminder about Lagrangian systems

Lagrangian systems play a central role in dynamical systems and physics, in particular for classical mechanics. We refer to [2] for more details.

Definition 14. An admissible Lagrangian function L is a function $\mathrm{L} : \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^d \to \mathbb{C}$ such that $\mathrm{L}(t, x, v)$ is holomorphic with respect to v, differentiable with respect to x and real when $v \in \mathbb{R}$, continuous with respect to t.

A Lagrangian function defines a *functional* on $\mathcal{C}^1(I, \mathbb{R})$, denoted by

$$\mathcal{L}_{a,b} : \mathcal{C}^1\big(I, \mathbb{R}^d\big) \to \mathbb{R}, \quad x \in \mathcal{C}^1\big(I, \mathbb{R}^d\big) \mapsto \int_a^b \mathrm{L}\left(s, x(s), \frac{dx}{dt}(s)\right) ds. \tag{22}$$

When no confusion is possible we will simply write $\mathcal{L}$ for $\mathcal{L}_{a,b}$.

The classical *calculus of variations* analyzes the behavior of $\mathcal{L}$ under small perturbations of the initial function x. The main ingredient is a notion of differentiable functional and extremal.

Definition 15 *(Space of variations).* We denote by $V(a, b)$ the set of functions h in $\mathcal{C}^1(I, \mathbb{R}^d)$ such that $h(a) = h(b) = 0$.

A functional $\mathcal{L}$ is *$V(a,b)$-differentiable* at point $\gamma \in \mathcal{C}^1(I, \mathbb{R}^d)$ if and only if

$$\mathcal{L}(\gamma + \theta h) - \mathcal{L}(\gamma) = \theta D\mathcal{L}(\gamma)(h) + o(\theta),$$

for $\theta > 0$ sufficiently small and any $h \in V(a, b)$.

Definition 16. An extremal for the functional $\mathcal{L}$ is a function $\gamma \in \mathcal{C}^1(I, \mathbb{R}^d)$ such that $D\mathcal{L}(\gamma)(h) = 0$ for any $h \in V(a, b)$, where $D\mathcal{L}(\gamma)(h)$ is the Fréchet derivative of $\mathcal{L}$ at point γ in the direction h.

Extremals of the functional $\mathcal{L}$ can be characterized by an ordinary differential equation of order 2, called the Euler–Lagrange equation.

Theorem 2. *The extremals of $\mathcal{L}$ coincide with the solutions of the Euler–Lagrange equation denoted by* (*EL*) *and defined by*

$$\frac{d}{dt}\left[\frac{\partial \mathrm{L}}{\partial v}\left(t, \gamma(t), \frac{d\gamma}{dt}(t)\right)\right] = \frac{\partial \mathrm{L}}{\partial x}\left(t, \gamma(t), \frac{d\gamma}{dt}(t)\right). \tag{EL}$$

This equation can be seen as the action of the differential operator

$$\mathrm{O}_{(EL)} = \frac{d}{dt} \circ \frac{\partial \mathrm{L}}{\partial v} - \frac{\partial \mathrm{L}}{\partial x} \tag{23}$$

on $(t, \gamma(t), \frac{d\gamma}{dt}(t))$. The Euler–Lagrange equation (*EL*) is then

$$\mathrm{O}_{(EL)}\left(t, \gamma(t), \frac{d\gamma}{dt}(t)\right) = 0.$$

2.1.2. Non-differentiable Euler–Lagrange equation

The non-differentiable embedding procedure allows us to define a natural extension of the classical Euler–Lagrange equation in the non-differentiable context.

Lemma 5. *Let L be an admissible Lagrangian function. The non-differentiable embedding of the Euler–Lagrange differential operator* $\mathrm{O}_{(EL)}$ *is given by*

$$\mathrm{Emb}_{\Box}(\mathrm{O}_{(EL)}) = \frac{\Box}{\Box t} \circ \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x}. \tag{24}$$

Proof. The operator (23) is first considered as acting on $\mathcal{C}^1(I,\mathbb{R}) \times \mathcal{C}^1(I,\mathbb{R}^d) \times \mathcal{C}^0(I,\mathbb{C}^d)$, *i.e.* for all $(t, x(t), y(t)) \in I \times \mathcal{C}^1(I,\mathbb{R}^d) \times \mathcal{C}^0(I,\mathbb{C}^d)$ we have

$$\mathrm{O}_{(EL)}\big(t, x(t), y(t)\big) = \frac{d}{dt}\left(\frac{\partial L}{\partial v}\big(t, x(t), y(t)\big)\right) - \frac{\partial L}{\partial x}\big(t, x(t), y(t)\big).$$

This operator is of the form $\mathrm{O}_{\mathbf{f}}^{\mathbf{g}}$ with

$$\mathbf{f} = \left(-\frac{\partial L}{\partial x}, \mathbf{1}\right),$$

and

$$\mathbf{g} = \left(\mathbf{1}, \frac{\partial L}{\partial v}\right),$$

where $\mathbf{1} : \mathbb{R} \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$ is the *constant* function $\mathbf{1}(t, x, y) = 1$. As a consequence, $\mathrm{O}_{(EL)}$ is given by

$$\mathrm{O}_{(EL)} = \mathbf{1} \cdot \frac{d}{dt} \circ \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} \cdot \mathrm{Id} \circ \mathbf{1},$$

with the convention that $(\frac{d}{dt})^0 = \mathrm{Id}$. We then obtain Eq. (24) using Definition 11. $\Box$

Remark 5. Another possible choice would be $\mathbf{f} = (\mathbf{1}, \mathbf{1})$ and $\mathbf{g} = (-\frac{\partial L}{\partial v}, \frac{\partial L}{\partial x})$, *i.e.* exchanging the f_0 and g_0 terms. This alternative is always possible, but we use the convention to put the dependence on f_0 because the usual form of a differential operator is $\sum_i a_i \frac{d^i}{dt^i}$ which can be written only with a component $\mathbf{f}$. Moreover, taking this alternative for the 0-term in $\mathbf{f}$ and $\mathbf{g}$ does not affect the resulting form of the embedded equation (this is of course not the case when dealing with i-th terms, $i \geqslant 1$).

Theorem 3. *Let L be an admissible Lagrangian function. The non-differentiable embedded Euler–Lagrange equation, denoted by* Emb(*EL*) *associated to L is given by*

$$\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}\left(t, \gamma(t), \frac{\Box}{\Box t}\gamma(t)\right)\right) - \frac{\partial L}{\partial x}\left(t, \gamma(t), \frac{\Box}{\Box t}\gamma(t)\right) = 0. \tag{Emb(EL)}$$

Proof. Using Definition 13, the non-differentiable embedding of equation (*EL*) is given by

$$\mathrm{Emb}_{\Box}(\mathrm{O}_{(EL)})\left(\bullet, x(\bullet), \frac{\Box}{\Box t}x(\bullet)\right) = 0,$$

which reduces to equation Emb(*EL*) thanks to Lemma 5. $\Box$

2.2. Embedding of Lagrangian systems

In this subsection, we derive the non-differentiable embedding of a particular class of ordinary differential equations called *Euler–Lagrange equations* which governs the dynamics of *Lagrangian systems*.

2.2.1. Embedding of the Lagrangian functional

Definition 17. Let a Lagrangian functional $\mathcal{L}_{a,b}$ be given as defined in (22). The natural embedding of the Lagrangian functional $\mathcal{L}_{a,b}$ is given by

$$\mathcal{L}_{\Box} : \mathcal{C}^1_{\Box}\big(I, \mathbb{R}^d\big) \to \mathbb{R}, \quad x \in \mathcal{C}^1_{\Box}\big(I, \mathbb{R}^d\big) \mapsto \int_a^b \mathrm{L}\left(s, x(s), \frac{\Box x(s)}{\Box t}\right) ds. \tag{25}$$

2.2.2. Non-differentiable calculus of variations

Let α be a real number $0 < \alpha < 1$ and ϵ be a parameter which is assumed to be sufficiently small, *i.e.* $0 < \epsilon \ll 1$, without specifying its exact smallness.

Definition 18. Let $\gamma \in H^\alpha(I, \mathbb{R}^d)$. Let V be a subvectorial space of $H_0^\beta := \{h \in H^\beta(I, \mathbb{R}^d),\ h(a) = h(b) = 0\}$, with $\alpha + \beta > 1$, the space of non-differentiable variations. A V-variation γ' of γ is a curve in $\mathcal{C}^0(I, \mathbb{R}^d)$ defined by

$$\gamma'(t) := \gamma(t) + h(t), \quad h \in V.$$

Such a curve is denoted by $\gamma' := \gamma + h$.

Definition 19. Let $\Phi : \mathcal{C}^0(I, \mathbb{R}^d) \to \mathbb{C}$ be a functional. The functional Φ is called *V-differentiable* on a curve $\gamma \in H^\alpha(I, \mathbb{R}^d)$ if and only if its Fréchet differential

$$\lim_{\epsilon \to 0} \frac{\Phi(\gamma + \epsilon h) - \Phi(\gamma)}{\epsilon}$$

exists in any direction $h \in V$. And then $D\Phi$ is called its differential and is given by

$$D\Phi(\gamma)(h) = \lim_{\epsilon \to 0} \frac{\Phi(\gamma + \epsilon h) - \Phi(\gamma)}{\epsilon}.$$

Definition 20 *(V-extremal curves).* A V-extremal curve of the functional Φ is a curve $\gamma \in H^\alpha(I, \mathbb{R}^d)$ satisfying

$$D\Phi(\gamma)(h) = 0,$$

for any $h \in V$.

The following theorem gives the analogous of the Euler–Lagrange equations for extremals of our functionals.

Theorem 4. *The differential of $\mathcal{L}_\Box$ on $\gamma \in H^\alpha(I, \mathbb{R}^d) \cap \mathcal{C}^1_\Box(I, \mathbb{R}^d)$ is given by*

$$D\mathcal{L}_\Box(\gamma)(h) = \int_a^b \left(\frac{\partial L}{\partial x}\left(t, \gamma(t), \frac{\Box \gamma(t)}{\Box t}\right) \cdot h(t) + \frac{\partial L}{\partial v}\left(t, \gamma(t), \frac{\Box \gamma(t)}{\Box t}\right) \cdot \frac{\Box h(t)}{\Box t} \right) dt, \tag{26}$$

for any $h \in V$.

Proof. Let $\Phi : H^\alpha(I, \mathbb{R}^d) \to \mathbb{C}$ be the functional defined by

$$\Phi(\gamma) = \int_a^b L\left(t, \gamma(t), \frac{\Box \gamma(t)}{\Box t}\right) dt, \tag{27}$$

for any $\gamma \in H^\alpha(I, \mathbb{R}^d)$. With help of a Taylor expansion, we obtain its differential given by:

$$D\Phi(\gamma)(h) = \int_a^b \left(\frac{\partial L}{\partial x}\left(t, \gamma(t), \frac{\Box \gamma(t)}{\Box t}\right) h(t) + \frac{\partial L}{\partial v}\left(t, \gamma(t), \frac{\Box \gamma(t)}{\Box t}\right) \frac{\Box h(t)}{\Box t} \right) dt. \qquad \Box$$

Theorem 5 *(Non-differentiable least-action principle). Let $0 < \alpha < 1$, $\alpha + \beta > 1$. Let L be an admissible Lagrangian function of class $\mathcal{C}^2$. We assume that $\gamma \in H^\alpha(I, \mathbb{R}^d)$, such that $\frac{\Box \gamma}{\Box t} \in H^\alpha(I, \mathbb{R}^d)$ and $\frac{\partial L}{\partial v}(t, \gamma, \Box_t \gamma).h$ satisfies condition (7) for all $h \in H_0^\beta(I, \mathbb{R}^d)$. A curve $\gamma \in H^\alpha(I, \mathbb{R}^d)$ satisfying the following generalized Euler–Lagrange equation*

$$\frac{\partial L}{\partial x}\left(t, \gamma(t), \frac{\Box \gamma(t)}{\Box t}\right) - \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}\left(t, \gamma(t), \frac{\Box \gamma(t)}{\Box t}\right)\right) = 0 \tag{NDEL}$$

is an extremal curve of the functional (25) on the space of variations $V = H_0^\beta(I, \mathbb{R}^d)$.

Proof. As $\frac{\partial L}{\partial v}(t, \gamma(t), \frac{\Box \gamma(t)}{\Box t})$ is $H^\alpha(I, \mathbb{R}^d)$, and $h \in V = H_0^\beta(I, \mathbb{R}^d)$, with $\alpha + \beta > 1$, we obtain using Lemma 1

$$\int_a^b \frac{\partial L}{\partial v} \cdot \frac{\Box h(t)}{\Box t}\, dt = \int_a^b \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v} \cdot h\right) dt - \int_a^b \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}\right) \cdot h.$$

As $\frac{\partial L}{\partial v}.h$ satisfies condition (7) for all $h \in V$, we obtain using Lemma 2 and that $h(a) = h(b) = 0$

$$\int_a^b \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}.h\right) dt = 0.$$

The differential (26) becomes

$$D\mathcal{L}_\Box(\gamma)(h) = \int_a^b \left[\frac{\partial L}{\partial x}\left(t, \gamma(t), \frac{\Box\gamma(t)}{\Box t}\right) - \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}\left(t, \gamma(t), \frac{\Box\gamma(t)}{\Box t}\right)\right)\right] \cdot h(t)\, dt, \tag{28}$$

for all $h \in V = H_0^\beta$. $\Box$

2.3. Coherence problem

We prove that the non-differentiable embedding is coherent, *i.e.* that the embedded Euler–Lagrange equation coincides with the non-differentiable Euler–Lagrange equation obtained using the non-differentiable calculus of variations. We discuss the compatibility between the non-differentiable embedding of Lagrangian systems and the non-differentiable calculus of variations using the notion of coherent embedding.

Definition 21. An embedding procedure is called *coherent* when the two Euler–Lagrange equations are the same, *i.e.* if

$$\mathrm{NDEL} = \mathrm{Emb}(EL),$$

assuming that NDEL is obtained from the embedding of the classical functional using the *same* embedding procedure.

We have the following result:

Theorem 6 *(Coherence principle). Let L be an admissible Lagrangian function, then the following diagram commutes*

$$\begin{tikzcd} \mathcal{L}(x) \arrow[r, "\mathrm{Emb}_\Box"] \arrow[d, "L.A.P."'] & \mathcal{L}_\Box(x) \arrow[d, "N.D.L.A.P."] \\ \frac{d}{dt}(\frac{\partial L}{\partial v}(z(t))) = \frac{\partial L}{\partial x}(z(t)) \arrow[r, "\mathrm{Emb}_\Box"] & \frac{\Box}{\Box t}(\frac{\partial L}{\partial v}(Z(t))) = \frac{\partial L}{\partial x}(Z(t)) \end{tikzcd} \tag{29}$$

where $z(t) = (x(t), \dot{x}(t))$, $Z(t) = (x(t), \frac{\Box x(t)}{\Box t})$, and L.A.P. stands for "Least-Action Principle" and N.D.L.A.P. for the Non-Differentiable Least-Action Principle.

Remark 6. An embedding procedure is not always coherent. We refer to [9] and [7] where the coherence problem is discussed for the stochastic and the fractional embedding respectively.

This result gives a strong support to the non-differentiable embedding of Lagrangian equations. Indeed, we proved that the non-differentiable Euler–Lagrange equation derives from an extended variational principle constructed on the non-differentiable embedding of the classical functional.

In the following section, we prove that the solutions of Schrödinger equation as well as the ones of Navier–Stokes equation can be seen as solutions of a non-differentiable Euler–Lagrange equation.

3. Non-differentiable Hamiltonian systems

3.1. Reminder about Hamiltonian systems

Let L be an admissible Lagrangian function. If L satisfies the so-called Legendre property, we can associate to L, a Hamiltonian function denoted by H. Indeed, the basic idea underlying the Hamiltonian formalism is to code the dichotomy between speed and position.

Definition 22. Let L be an admissible Lagrangian function. The Lagrangian L is said to satisfy the Legendre property if the mapping $v \mapsto \frac{\partial L}{\partial v}(t, x, v)$ is invertible for any $(t, x, v) \in I \times \mathbb{R}^d \times \mathbb{C}^d$.

As a consequence, if we introduce a new variable

$$p = \frac{\partial L}{\partial v}(t, x, v)$$

and L satisfies the Legendre property we can find a function f such that

$$v = f(t, x, p).$$

Using this notation, we have the following definition.

Definition 23. Let L be an admissible Lagrangian function satisfying the Legendre property. The Hamiltonian function H associated to L, is given by

$$\begin{aligned} H : \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^d &\to \mathbb{C} \\ (t, x, p) &\mapsto H(t, x, p) = pf(t, x, p) - L\big(t, x, f(t, x, p)\big). \end{aligned}$$

A natural Lagrangian function is usually of the form

$$L(t, x, v) = \frac{1}{2}mv^2 - U(x),$$

where $m > 0$. The term $\frac{1}{2}mv^2$ is the kinetic energy and $U(x)$ the potential energy. Then the Hamiltonian function associated to L is

$$H(t, x, p) = \frac{1}{2m}p^2 + U(x)$$

and represents the total energy of the system.

The dynamics generated by a Hamiltonian system is defined as follows:

Proposition 1 *(Hamilton's least-action principle). (See [2].) A curve* $(x, p) \in \mathcal{C}^1(I, \mathbb{R}^d) \times \mathcal{C}^1(I, \mathbb{C}^d)$ *is an extremal of the Hamiltonian functional*

$$\mathcal{H}(x, p) = \int_a^b p(t)\,\frac{dx}{dt}(t) - H\big(t, x(t), p(t)\big)\,dt \tag{30}$$

if and only if it satisfies the Hamiltonian system associated to H given by

$$\begin{cases} \dfrac{dx}{dt}(t) = \dfrac{\partial H}{\partial p}\big(t, x(t), p(t)\big), \\ \dfrac{dp}{dt}(t) = -\dfrac{\partial H}{\partial x}\big(t, x(t), p(t)\big). \end{cases} \tag{31}$$

3.2. The non-differentiable case

The non-differentiable embedding induces a change in the phase space with respect to the classical case. As a consequence, we have to work with variables (x, p) which belong to $\mathbb{R}^d \times \mathbb{C}^d$ and not only to $\mathbb{R}^d \times \mathbb{R}^d$ as usual. This means that we must embed the Hamiltonian system in $\mathcal{C}^0(I, \mathbb{R}^d) \times \mathcal{C}^0(I, \mathbb{C}^d)$.

Lemma 6.

1. *The embedded Hamiltonian system* (31) *is given by*:

$$\begin{cases} \dfrac{\Box x}{\Box t}(t) = \dfrac{\partial H}{\partial p}\big(t, x(t), p(t)\big), \\ \dfrac{\Box p}{\Box t}(t) = -\dfrac{\partial H}{\partial x}\big(t, x(t), p(t)\big). \end{cases} \tag{32}$$

2. *The embedded Hamiltonian functional* $\mathcal{H}_\Box$ *is defined on* $H^\alpha(I, \mathbb{R}^d) \times H^\alpha(I, \mathbb{C}^d)$ *by*:

$$\mathcal{H}_\Box(x, p) = \int_a^b p(t)\frac{\Box x(t)}{\Box t} - H\big(t, x(t), p(t)\big)\,dt.$$

Using the non-differentiable calculus of variations we can derive the Euler–Lagrange equation for $\mathcal{H}_\square$.

Theorem 7 *(Non-differentiable Hamilton's least-action principle). Let $0 < \alpha < 1$, $\alpha + \beta > 1$. Let L be an admissible Lagrangian function of class $\mathcal{C}^2$ satisfying the Legendre property. We assume that $\gamma \in H^\alpha(I, \mathbb{R}^d)$, such that $\frac{\square\gamma}{\square t} \in H^\alpha(I, \mathbb{R}^d)$ and $\frac{\partial L}{\partial v}(t, \gamma, \square\gamma).h$ satisfies condition (7) for all $h \in H_0^\beta(I, \mathbb{R}^d)$.*

Let H be the corresponding Hamiltonian defined by (30). A curve $\gamma : t \to (t, x(t), p(t)) \in I \times \mathbb{R}^d \times \mathbb{C}^d$ solution of the non-differentiable Hamiltonian system (32) is an extremal of the functional

$$\mathcal{H}_\square(x, p) = \int_a^b p(t)\frac{\square x(t)}{\square t} - H\big(t, x(t), p(t)\big)\,dt,$$

over $V = H_0^\beta(I; \mathbb{R}^d) \times H_0^\beta(I; \mathbb{C}^d)$.

Proof. The functional $\mathcal{H}_\square$ is a Lagrangian functional with Lagrangian $\mathbb{L}$ given by

$$\mathbb{L} : \begin{array}{l} \mathbb{R}^d \times \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}, \\ \qquad (t, x, p, v, w) \mapsto \mathbb{L}(t, x, p, v, w) = pv - H(t, x, p), \end{array}$$

the Lagrangian being evaluated on $(t, x(t), p(t), \frac{\square x}{\square t}, \frac{\square p}{\square t})$. As $(x, p) \in H^\alpha(I, \mathbb{R}^d) \times H^\alpha(I, \mathbb{C}^d)$, $0 < \alpha < 1$, the non-differentiable Euler–Lagrange equation is given by

$$\begin{cases} \dfrac{\square}{\square t}\left(\dfrac{\partial \mathbb{L}}{\partial v}\left(t, x, p, \dfrac{\square x}{\square t}, \dfrac{\square p}{\square t}\right)\right) = \dfrac{\partial \mathbb{L}}{\partial x}\left(t, x, p, \dfrac{\square x}{\square t}, \dfrac{\square p}{\square t}\right), \\ \dfrac{\square}{\square t}\left(\dfrac{\partial \mathbb{L}}{\partial w}\left(t, x, p, \dfrac{\square x}{\square t}, \dfrac{\square p}{\square t}\right)\right) = \dfrac{\partial \mathbb{L}}{\partial p}\left(t, x, p, \dfrac{\square x}{\square t}, \dfrac{\square p}{\square t}\right), \end{cases}$$

which leads to

$$\begin{cases} \dfrac{\square p}{\square t} = -\dfrac{\partial H}{\partial x}, \\ 0 = \dfrac{\square x}{\square t} - \dfrac{\partial H}{\partial p}. \end{cases}$$

By Theorem 5 a solution of the non-differentiable Euler–Lagrange equation for $\mathbb{L}$ is an extremal of the associated functional $\mathcal{H}_\square$. As these solutions are given by the non-differentiable Hamiltonian system, this concludes the proof. $\square$

Again we have coherence of our embedding procedure with respect to the Hamiltonian formalism.

Corollary 2 *(Coherence). Let H be a Hamiltonian function, then the following diagram commutes*

$$\begin{tikzcd} \mathcal{H}(x, p) \arrow[r, "\mathrm{Emb}_\square"] \arrow[d, "L.A.P."'] & \mathcal{H}_\square(x, p) \arrow[d, "N.D.L.A.P."] \\ \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial p}, \\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial x}, \end{cases} \arrow[r, "\mathrm{Emb}_\square"] & \begin{cases} \dfrac{\square x}{\square t} = \dfrac{\partial H}{\partial p}, \\ \dfrac{\square p}{\square t} = -\dfrac{\partial H}{\partial x}. \end{cases} \end{tikzcd} \tag{33}$$

4. Application to PDEs

4.1. Non-differentiable method of characteristics

The classical method of characteristics for a PDE is to look for curves $s \mapsto (x(s), t(s))$ where $x(s)$ and $t(s)$ are solutions of an ordinary differential equation such that solutions $u(x, t)$ of the PDE satisfies

$$\frac{d}{ds}\big(u\big(x(s), t(s)\big)\big) = F\big(x(s), t(s)\big),$$

where F is the non-homogeneous part of the PDE.

In many cases, we can choose

$$\frac{dt}{ds} = 1$$

so that one is reduced to find a curve $t \to x(t)$ satisfying the following ordinary differential equation

$$\frac{d}{dt}\big(u\big(x(t), t\big)\big) = F\big(x(t), t\big).$$

The method of characteristics does not work for parabolic PDEs and PDEs of mixed type like hyperbolic/parabolic (as for example the transport equation with diffusion).

Using the operator $\frac{\Box}{\Box t}$ one can generalize this method. We say that a curve $s \to (x(s), t(s))$ is a non-differentiable characteristic for a given PDE if the solution $u(x,t)$ satisfies

$$\frac{\Box}{\Box s}\big(u\big(x(s), t(s)\big)\big) = F\big(x(s), t(s)\big),$$

and x and t satisfy an ordinary differential equation in $\frac{\Box}{\Box t}$.

In the following subsection, we characterize the non-differentiable characteristics of the Navier–Stokes equations.

4.2. Non-differentiable characteristics for the Navier–Stokes equation

There exist already known tentative for deriving the equations of fluids mechanics from a variational principle. We refer in particular to the work of Arnold [1] and [10,3] which deal with an interpretation of the Euler equation as geodesics on an infinite Lie group. For an overview of results and questions in fluid mechanics we refer to [11,18].

In this subsection, we characterize non-differentiable characteristics for the Navier–Stokes equation as critical point of a non-differentiable Lagrangian functional.

The incompressible homogeneous Navier–Stokes equation looks like

$$\begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{k=1}^{d} u_k \frac{\partial u}{\partial x_k} = \nu \Delta_x u - \nabla_x p, \\ \operatorname{div} u = 0, \end{cases} \tag{34}$$

where the unknown are the velocity $u(t,x) \in \mathbb{R}^d$, $u = (u_1, \ldots, u_d)$, and the pressure $p(t,x) \in \mathbb{R}$. The constant $\nu \in \mathbb{R}^+$ is the viscosity. Eq. (34) is also equivalent to

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + \displaystyle\sum_{k=1}^{d} u_k \frac{\partial u_i}{\partial x_k} = \nu \Delta_x u_i - \frac{\partial p}{\partial x_i}, \quad i = 1, \ldots, d, \\ \operatorname{div} u = 0. \end{cases}$$

We look for a curve $t \to x(t)$ such that

$$\frac{\Box}{\Box t}\big(u\big(x(t), t\big)\big) = -\nabla_x p.$$

The solution u of Eq. (34) is such that $u \in \mathcal{C}^2(I \times \mathbb{R}^d, \mathbb{R})$, then if we take $x \in \mathcal{C}^{1,1/2}_{\lambda,\mathbf{W}}(I, \mathbb{R}^d)$ then from Corollary 1, formula (17), we have for any $i = 1, \ldots, d$

$$\frac{\Box u_i(x(t), t)}{\Box t} = \nabla_x u_i\big(x(t), t\big) \cdot \frac{\Box x}{\Box t}(t) + \frac{\partial u_i}{\partial t}\big(x(t), t\big) - \frac{\lambda^2}{2} \Delta u_i\big(x(t), t\big). \tag{35}$$

In order to obtain $-\nabla_x p$ we must choose $\Box x / \Box t$ and λ in (35) such that

$$\nabla_x u_i\big(x(t), t\big) \cdot \frac{\Box x}{\Box t}(t) + \frac{\partial u_i}{\partial t}\big(x(t), t\big) - \frac{\lambda^2}{2} \Delta u_i\big(x(t), t\big) = \frac{\partial u_i}{\partial t} + \sum_{k=1}^{d} u_k \frac{\partial u_i}{\partial x_k} - \nu \Delta_x u_i, \quad i = 1, \ldots, d.$$

We then consider $x = (x_1, \ldots, x_d) \in \mathcal{C}^{1,1/2}_{\lambda,\mathbf{W}}(I, \mathbb{R}^d)$ of the form

$$x_i(t) = \int_0^t u_i\big(x(s), s\big)\, ds + \lambda \mathbf{W}_i(t), \quad i = 1, \ldots, d. \tag{36}$$

Then, $\frac{\Box x}{\Box t}(t) = u(x(t), t)$. As a consequence, we obtain for any $i = 1, \ldots, d$

$$\frac{\Box u_i(x(t),t)}{\Box t} = \nabla_x u_i\big(x(t),t\big)\cdot u\big(x(t),t\big) + \frac{\partial u_i}{\partial t}\big(x(t),t\big) - \frac{\lambda^2}{2}\Delta u_i\big(x(t),t\big).$$

By fixing $\lambda^2 = 2\nu$, *i.e.* working on $\mathcal{C}^{1,1/2}_{\sqrt{2\nu},\mathbf{W}}(I,\mathbb{R}^d)$ and since u satisfies the Navier–Stokes equation, we obtain $\frac{\Box}{\Box t}(u_i(x(t),t)) = -\frac{\partial p}{\partial x_i}$, *i.e.* that x is a non-differentiable characteristic for the Navier–Stokes equation.

We then introduce the following space:

Definition 24. We denote by $\mathcal{C}_{\mathrm{nav}}$ the subset of $C^{1,1/2}_{\sqrt{2\nu},\mathbf{W}}(I;\mathbb{R}^d)$ defined by:

$$\mathcal{C}_{\mathrm{nav}} := \left\{ x = (x_1,\ldots,x_d) \in \mathcal{C}^{1,1/2}_{\sqrt{2\nu},\mathbf{W}}\big(I,\mathbb{R}^d\big),\ x_i(t) = \int_0^t u_i\big(x(s),s\big)\,ds + \sqrt{2\nu}\mathbf{W}_i(t),\ i=1,\ldots,d \right\},$$

where u is a solution of the Navier–Stokes equation.

Let us note that on $\mathcal{C}_{\mathrm{nav}}$ the non-differentiable characteristics satisfy by definition

$$\frac{\Box}{\Box t}\big(u\big(x(t),t\big)\big) = -\nabla_x p, \tag{37}$$

which can be rewritten as

$$\frac{\Box}{\Box t}\left(\frac{\Box x}{\Box t}\right) = -\nabla_x p. \tag{38}$$

This equation looks like a non-differentiable Euler–Lagrange equation. In fact, one can prove that non-differentiable characteristics of the Navier–Stokes equation correspond to critical point of a non-differentiable Lagrangian functional.

Theorem 8. *Let L be the Lagrangian*

$$L(t,x,v) = \frac{1}{2}v^2 - p(x,t), \tag{39}$$

where p is the pressure.

We assume that for any $x \in \mathcal{C}_{\mathrm{nav}}$ and $h \in H^\beta_0(I;\mathbb{R}^d)$, $\beta > 1/2$, the function $t \mapsto u(x(t),t)h(t)$ satisfies condition (7)*. Then the non-differentiable characteristics $x \in \mathcal{C}_{\mathrm{nav}}$ of the Navier–Stokes equations correspond to $H^\beta_0(I;\mathbb{R}^d)$, $\beta > 1/2$ extremals of the Lagrangian* (39)*.*

Proof. Theorem 5 adapted to functions in $\mathcal{C}_{\mathrm{nav}}$, gives the non-differentiable extremals of (39) over $\mathcal{C}_{\mathrm{nav}}$ by

$$\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}\left(t,x,\frac{\Box x}{\Box t}\right)\right) = \frac{\partial L}{\partial x}\left(t,x,\frac{\Box x}{\Box t}\right), \quad x \in \mathcal{C}_{\mathrm{nav}}.$$

As $\frac{\partial L}{\partial v} = v$ and $\frac{\partial L}{\partial x} = -\nabla_x p$, we obtain (38) which coincides with (37) over $\mathcal{C}_{\mathrm{nav}}$. $\Box$

We have now a clear understanding of the different roles of the pressure and the viscosity terms:

– the pressure $p(x)$ plays the role of a potential for the underlying classical dynamics,
– the viscosity ν controls the irregularity of solutions via $\mathbf{W}$.

It is not easy to characterize the set $\mathcal{C}_{\mathrm{nav}}$ and more work is needed in this direction. We only remark that this condition, on the quadratic part of the velocity can be realized in the stochastic setting using diffusion processes with constant diffusion.

4.3. The Schrödinger equation

In [5], we already proved that we can recover the Schrödinger equation in dimension one using an ϵ-dependent embedding (see Remark 3) and a non-differentiable variational principle. This is not satisfying due to the dependence on ϵ. Using the new framework we defined in Section 2, we prove that the solutions of the Schrödinger equation in $\mathbb{R}^d$, $d \geqslant 1$, can be seen as extremals of the non-differentiable embedded Newton's Lagrangian in $\mathcal{C}^{1,1/2}_{\lambda,\mathbb{W}}(I,\mathbb{R}^d)$ under specific constraints on λ and the regular part.

In the following, we denote by ψ an application

$$\psi: \begin{array}{l} \mathbb{R}^d \times \mathbb{R} \to \mathbb{C} \\ (x,t) \mapsto \psi(x,t). \end{array}$$

Definition 25. We denote by $\mathcal{C}_{\mathrm{schr}}(\gamma,\lambda) \subset \mathcal{C}^{1,1/2}_{\lambda,\mathbb{W}}(I,\mathbb{R}^d)$ the set of $x \in \mathcal{C}^{1,1/2}_{\lambda,\mathbb{W}}(I,\mathbb{R}^d)$ such that

$$\frac{\Box x_j}{\Box t}(t) = -2i\gamma \frac{\partial \ln\psi}{\partial x_j}\big(x(t),t\big), \quad j=1,\dots,d, \tag{40}$$

where $\gamma \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, and $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ is $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}, \mathbb{C})$.

Let h be the Planck constant, m the mass of a particle, $\bar{h} = h/2\pi$ the reduced Planck constant. The dynamics of a particle of mass m in $\mathbb{R}^d$, $d \geqslant 1$, under the potential $U : \mathbb{R}^d \to \mathbb{R}$ in quantum mechanics is governed by the *Schrödinger equation*:

$$i\bar{h}\frac{\partial\psi}{\partial t} + \frac{\bar{h}^2}{2m}\sum_{k=1}^{d}\frac{\partial^2\psi}{\partial x_k^2} = U(x)\psi, \tag{41}$$

where $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ is the *wave function* associated to the particle.

Theorem 9. *Let L be the classical Newton's Lagrangian with potential U defined by*

$$L(t,x,v) = \frac{1}{2}mv^2 - U(x). \tag{42}$$

We assume that for any $x \in \mathcal{C}_{\mathrm{schr}}(\bar{h}/2m,\lambda)$ where $\lambda^2 = -i\frac{\bar{h}}{m}$ and $h \in H_0^\beta(I;\mathbb{R}^d)$, $\beta > 1/2$, the function $t \mapsto \frac{\Box x}{\Box t}(t)h(t)$ satisfies condition (7)*.*

Then the solutions of the Schrödinger equation (41) *coincide on $\mathcal{C}_{\mathrm{schr}}(\bar{h}/2m,\lambda)$ where $\lambda^2 = -i\frac{\bar{h}}{m}$ with extremals of the Newton's Lagrangian.*

Proof. By Theorem 5, extremals of our functional satisfy the non-differentiable Euler–Lagrange equation

$$\frac{\Box}{\Box t}\left(m\frac{\Box x_j(t)}{\Box t}\right) = -\frac{\partial U}{\partial x_j}(x), \quad j=1,\dots,d. \tag{43}$$

Since $x \in \mathcal{C}_{\mathrm{schr}}(\gamma,\delta)$, x satisfies (40):

$$\frac{\Box}{\Box t}\left[\frac{\Box x_j}{\Box t}\right] = -i2\gamma\frac{\Box}{\Box t}\left(\frac{\partial\ln(\psi)}{\partial x_j}\big(x(t),t\big)\right). \tag{44}$$

By Corollary 1, we have

$$\begin{aligned}\frac{\Box}{\Box t}\left[\frac{\Box x_j}{\Box t}\right] &= -i2\gamma\sum_{k=1}^{d}\frac{\Box x_k}{\Box t}\frac{\partial}{\partial x_k}\left(\frac{\partial\ln\psi}{\partial x_j}\right)\big(x(t),t\big) - i2\gamma\frac{\partial}{\partial t}\left(\frac{\partial\ln(\psi)}{\partial x_j}\right)\big(x(t),t\big) \\ &\quad - i\gamma\lambda^2\sum_{k=1}^{d}\frac{\partial^2}{\partial x_k^2}\left(\frac{\partial\ln\psi}{\partial x_j}\right)\big(x(t),t\big).\end{aligned} \tag{45}$$

Replacing $\frac{\Box x_k}{\Box t}$ by its expression as a function of ψ, we obtain

$$\begin{aligned}\frac{\Box x_k}{\Box t}\frac{\partial}{\partial x_k}\left(\frac{\partial\ln\psi}{\partial x_j}\right)\big(x(t),t\big) &= -i2\gamma\frac{\partial\ln\psi}{\partial x_k}\frac{\partial}{\partial x_k}\left(\frac{\partial\ln\psi}{\partial x_j}\right)\big(x(t),t\big) \\ &= -i\gamma\frac{\partial}{\partial x_j}\left[\frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial x_k}\right)^2\right]\big(x(t),t\big).\end{aligned} \tag{46}$$

Combining Eqs. (45) and (46), we get

$$\begin{aligned}\frac{\Box}{\Box t}\left[\frac{\Box x_j}{\Box t}\big(x(t),t\big)\right] &= -i2\gamma\frac{\partial}{\partial x_j}\left[-i\gamma\frac{1}{\psi^2}\sum_{k=1}^{d}\left(\frac{\partial\psi}{\partial x_k}\right)^2 + \frac{1}{\psi}\frac{\partial\psi}{\partial t} + \frac{\lambda^2}{2}\sum_{k=1}^{d}\left(\frac{1}{\psi}\frac{\partial^2\psi}{\partial x_k^2} - \frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial x_k}\right)^2\right)\right]\big(x(t),t\big) \\ &= -i2\gamma\frac{\partial}{\partial x_j}\left[-\left(i\gamma + \frac{\lambda^2}{2}\right)\frac{1}{\psi^2}\sum_{k=1}^{d}\left(\frac{\partial\psi}{\partial x_k}\right)^2 + \frac{1}{\psi}\frac{\partial\psi}{\partial t} + \frac{\lambda^2}{2}\frac{1}{\psi}\sum_{k=1}^{d}\frac{\partial^2\psi}{\partial x_k^2}\right]\big(x(t),t\big).\end{aligned}$$

The non-differentiable Euler–Lagrange equation (43) becomes

$$-i2\gamma m\frac{\partial}{\partial x_j}\left[-\left(i\gamma+\frac{\lambda^2}{2}\right)\frac{1}{\psi^2}\sum_{k=1}^{d}\left(\frac{\partial\psi}{\partial x_k}\right)^2+\frac{1}{\psi}\frac{\partial\psi}{\partial t}+\frac{\lambda^2}{2}\frac{1}{\psi}\sum_{k=1}^{d}\frac{\partial^2\psi}{\partial x_k^2}\right]=-\frac{\partial U}{\partial x_j}.$$

In order to recover the Schrödinger equation, one must choose δ and γ such that

$$-2i\gamma m=-i\bar{h},\qquad -i\gamma m\lambda^2=-\bar{h}^2/2m,\quad\text{and}\quad i\gamma+\frac{\lambda^2}{2}=0.$$

This system is overdetermined. The two first equations lead to

$$\gamma=\frac{\bar{h}}{2m},\qquad \lambda^2=-i\frac{\bar{h}}{m}.$$

We verify that the third constraint is satisfied.

As a consequence, for each $j=1,\dots,d$, we obtain

$$\frac{\partial}{\partial x_j}\left[-\frac{i\bar{h}}{\psi}\frac{\partial\psi}{\partial t}-\frac{\bar{h}^2}{2m}\frac{1}{\psi}\sum_{k=1}^{d}\frac{\partial^2\psi}{\partial x_k^2}+U\right]\bigl(x(t),t\bigr)=0,$$

meaning that the following equality is true

$$i\bar{h}\frac{\partial\psi}{\partial t}+\frac{\bar{h}^2}{2m}\sum_{k=1}^{d}\frac{\partial^2\psi}{\partial x_k^2}=U(x)\psi+c(x)$$

where $c(x)$ is an arbitrary function. This concludes the proof. $\square$

5. Notations

- $d\in\mathbb{N}$ is a fixed integer.
- I is an open set in $\mathbb{R}$.
- $a,b\in\mathbb{R}$, $a<b$, such that $[a,b]\subset I$.
- $\mathcal{F}(I,\mathbb{R}^d)$ is the set of functions from I to $\mathbb{R}^d$.
- $\mathcal{C}^0(I,\mathbb{R}^d)$ (respectively $\mathcal{C}^0(I,\mathbb{C}^d)$) is the set of continuous functions.
- $\mathcal{C}^1_\square(I,\mathbb{R})$ is the set of continuous functions $f\in\mathcal{C}^0(I,\mathbb{R})$ such that $\frac{\square f}{\square t}\in\mathcal{C}^0(I,\mathbb{R})$.
- $\mathcal{C}^n(I,\mathbb{R}^d)$ (respectively $\mathcal{C}^n(I,\mathbb{C}^d)$) is the set of functions in $\mathcal{C}^0(I,\mathbb{R}^d)$ (respectively $\mathcal{C}^0(I,\mathbb{C}^d)$) which are differentiable up to order n.
- W_α is a standard irregular function of order α.
- $\mathbb{W}$ (resp. $\mathbf{W}$) is a canonical (anti-canonical) irregular function.
- $\mathcal{C}^{k,\alpha}_{W_\alpha}$ is the subset of $\mathcal{C}^0(I,\mathbb{C}^d)$ defined by

$$\mathcal{C}^{k,\alpha}_{W_\alpha}=\bigl\{x:=u+\lambda.W_\alpha,\ u\in\mathcal{C}^k\bigl(I,\mathbb{C}^d\bigr),\ \lambda\in\mathbb{C}\bigr\}.$$

- Let $\lambda\in\mathbb{C}$, $\mathcal{C}^{k,\alpha}_{\lambda,W_\alpha}(I,\mathbb{C}^d)$ be the subset of $\mathcal{C}^{k,\alpha}_{W_\alpha}$ defined by

$$\mathcal{C}^{k,\alpha}_{\lambda,W_\alpha}=\bigl\{x:=u+\lambda.W_\alpha,\ u\in\mathcal{C}^k\bigl(I,\mathbb{C}^d\bigr)\bigr\}.$$

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